

The derivative of the natural logarithm function

This document shows how the formula for the derivative of the natural logarithm function, $\ln$, can be deduced from the formula for the derivative of the exponential function, $\exp$. The method is a special case of the proof of the *inverse function rule*, which is covered in Subsection 3.3 of Unit 7.

Since the natural logarithm function and the exponential function are inverses of each other, their graphs are the reflections of each other in the line $y = x$, as shown in Figure 1.

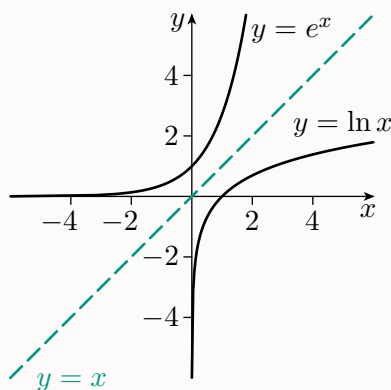


Figure 1 The graphs of $y = \ln x$ and $y = e^x$

So the gradients of the two graphs are related, which suggests that we ought to be able to use the formula for the derivative of the function $\exp$, which, as you saw in Unit 7, is

$$\frac{d}{dx}(e^x) = e^x,$$

to work out a formula for the derivative of the function $\ln$. Here's how we can do that.

The situation is that each tangent to the graph of $y = \ln x$ corresponds to a tangent to the graph of $y = e^x$, obtained by reflection in the line $y = x$, as illustrated in Figure 2.

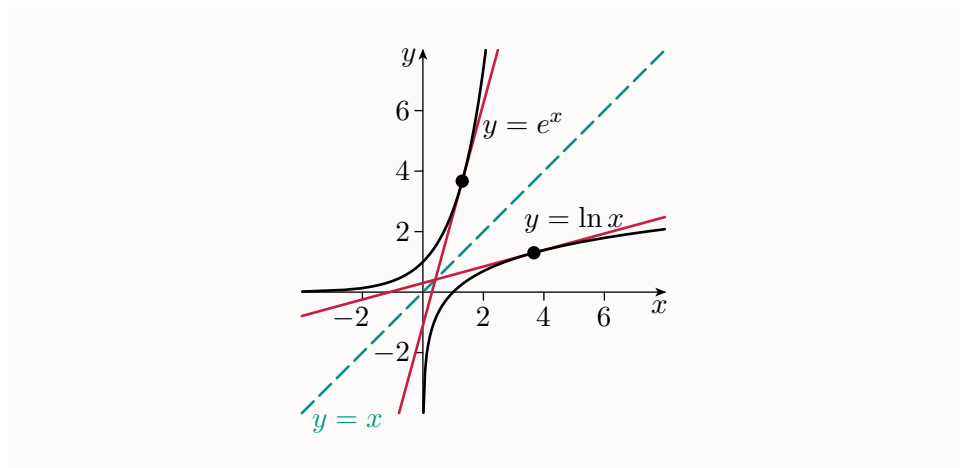


Figure 2 A tangent to the graph of $y = \ln x$ and its reflection in the line $y = x$, which is a tangent to the graph of $y = e^x$

So let's begin by working out how the gradients of two tangents like those in Figure 2 are related. To do this, think about any two straight lines, neither of which is vertical, that are reflections of each other in the line $y = x$. Consider any two points on one line, and the two points on the other line that are their reflections in the line $y = x$, as shown in Figure 3.

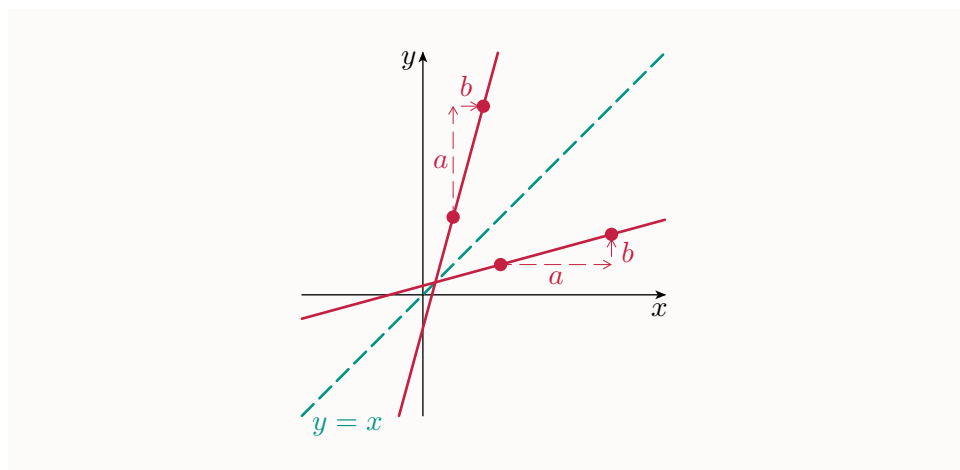


Figure 3 Two straight lines that are reflections of each other in the line $y = x$

Suppose that, for the pair of points on one line, run = a and rise = b , as shown. Then, for the pair of points on the other line, run = b and rise = a , as also shown. Hence the gradients of the two lines are b/a and a/b , respectively. So if two lines, neither of which is vertical, are reflections of each other in the line $y = x$, then their gradients are *reciprocals* of each other.

We can now use this fact, together with the formula for the derivative of the function $\exp$, to work out a formula for the derivative of the function $\ln$.

We have to consider a general point $(x, \ln x)$ on the graph of $y = \ln x$. The situation is illustrated in Figure 4. The reflection of the point $(x, \ln x)$ in the line $y = x$ has coordinates $(\ln x, x)$, since, as you saw in Unit 3, when you reflect a point in the line $y = x$ its x - and y -coordinates are interchanged. In particular, the reflected point has x -coordinate $\ln x$.

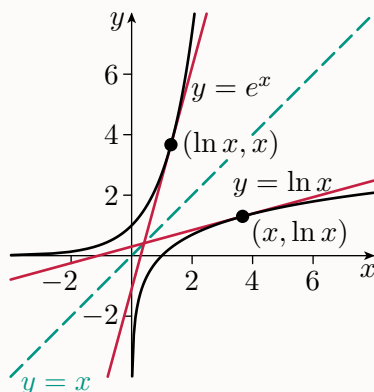


Figure 4 A point with x -coordinate x on the graph of $y = \ln x$, and its reflection in the line $y = x$

By the formula for the derivative of the exponential function, the gradient of the curve $y = e^x$ at the reflected point $(\ln x, x)$ is $e^{\ln x}$, which simplifies to just x . Hence the gradient of the curve $y = \ln x$ at the original point $(x, \ln x)$ is the reciprocal of x , that is, $1/x$. In other words, the formula for the derivative of the function $f(x) = \ln x$ is

$$f'(x) = \frac{1}{x}.$$

Since the function $f(x) = \ln x$ is defined only for $x \in (0, \infty)$ (and is differentiable at all values of x for which it is defined), the domain of its derivative is the interval $(0, \infty)$. So a complete specification of its derivative is

$$f'(x) = \frac{1}{x} \quad (x > 0).$$

As mentioned in Unit 7, you might like to compare the graphs of the functions $y = \ln x$ and $y = 1/x$ ($x > 0$), shown in Figure 5, and convince yourself that the function $y = 1/x$ ($x > 0$) does seem to give the gradients of the graph of the function $y = \ln x$. For example, the gradient of the graph of the function $y = \ln x$ is always positive, and indeed the values taken by the function $y = 1/x$ are always positive. Also, the gradient of the function $y = \ln x$ decreases as x increases, and indeed the values taken by the function $y = 1/x$ decrease as x increases.

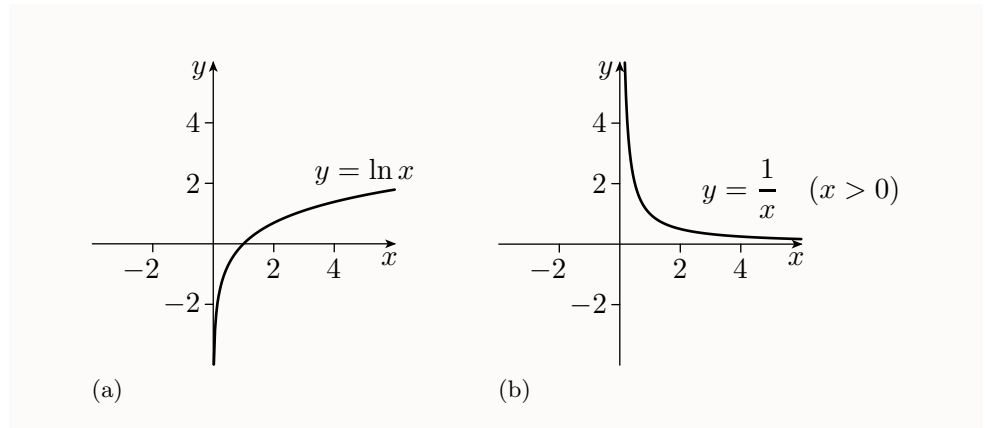


Figure 5 The graphs of (a) $y = \ln x$ (b) $y = 1/x$ for $x > 0$